

Constant volume exponential solutions in Einstein-Gauss-Bonnet flat anisotropic cosmology with a perfect fluid

Dmitry Chirkov

*Sternberg Astronomical Institute, Moscow State University, Moscow 119991 Russia and
Faculty of Physics, Moscow State University, Moscow 119991 Russia*

Sergey A. Pavluchenko

Instituto de Ciencias Físicas y Matemáticas, Universidad Austral de Chile, Valdivia, Chile

Alexey Toporensky

Sternberg Astronomical Institute, Moscow State University, Moscow 119991 Russia

In this paper we investigate the constant volume exponential solutions (i.e. the solutions with the scale factors change exponentially over time so that the comoving volume remains the same) in the Einstein-Gauss-Bonnet gravity. We find conditions for these solutions to exist and show that they are compatible with any perfect fluid with the equation of state parameter $\omega < 1/3$ if the matter density of the Universe exceeds some critical value. We write down some exact solutions which generalize ones found in our previous paper for models with a cosmological constant.

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I. INTRODUCTION

Exact solutions play important role in any gravitational theory, especially nonlinear. Indeed, using numerical receipts one almost always can build a solution, but its viability will be questioned. This is especially true for nonlinear theories where even numerical solutions are sometimes hard to find.

Lovelock gravity [1] is the striking example of the nonlinear theory of gravity. It is the most general metric theory of gravity yielding conserved second order equations of motion (in contrast to $f(R)$ gravity which gives fourth order dynamical equations) in arbitrary number of spacetime dimensions. One can say that the Lovelock gravity is a natural generalization of Einstein's General Relativity in the following sense: it is known [2–4] that the Einstein tensor is, in any dimension, the only symmetric and conserved tensor depending only on the metric and its first and second derivatives (with a linear dependence on second derivatives); if one drops the condition of linear dependence on second derivatives, one can obtain the most general tensor which satisfies other mentioned conditions – the Lovelock tensor.

The Lovelock gravity has been intensively studied in the cosmological context (see, e.g., [5–16]). Particularly, many interesting results have been obtained for flat anisotropic metrics due to the fact that its cosmological dynamics is much richer in the Lovelock gravity than in the Einstein one. Since the resulting

equations of motion turn out to be complicated enough, researchers usually study some special kind of metric (e.g. with only two different scale factors [13, 17]) or consider Lagrangians that contain the highest order Lovelock term only (e.g., deleting Einstein term and keeping Gauss-Bonnet term in a Lagrangian in the cases of (4+1)- and (5+1)-dimensional spacetimes one get so called “pure” Gauss-Bonnet model – see, for instance, [18–20]). In the latter approach solutions with power-law and exponential time dependence of scale factors were found. The first of them is an analog of Kasner solution [21, 22] – scale factors in this solution have power-law behavior, though relations between power indices is different from the Kasner solution in Einstein gravity [18, 23, 24]. Special features of the second type of solutions – Hubble parameters are constant, so in a flat metric differential equations of motion become algebraic – allows us to study them in more complicated theories [24, 25].

In our previous paper [26] we started to investigate the exponential solutions in Einstein-Gauss-Bonnet gravity. In the course of the study we have shown that these solution are divided into two different types – with constant volume and with volume changing in time. The paper [26] is devoted to the latter case. In the present paper we consider solutions with constant volume.

The structure of the manuscript is as follows: in the second section we introduce the set-up we are working on and very briefly reintroduce the results from our previous paper. Then in Section III we write down solutions of a special structure which generalize those found in [26] and in Section IV finally we work with a general case. Section V concludes the results of this paper and compares them with results of our previous paper [26].

II. THE SET-UP.

The Einstein-Gauss-Bonnet action in $(N + 1)$ -dimensional spacetime reads¹:

$$S = \frac{1}{16\pi} \int d^{N+1}x \sqrt{-g} (\mathcal{L}_E + \alpha \mathcal{L}_{GB} + \mathcal{L}_m), \quad \mathcal{L}_E = R, \quad \mathcal{L}_{GB} = R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} - 4R_{\alpha\beta} R^{\alpha\beta} + R^2, \quad (1)$$

where $R, R_{\alpha\beta}, R_{\alpha\beta\gamma\delta}$ are the $(N + 1)$ -dimensional scalar curvature, Ricci tensor and Riemann tensor respectively, α is the couple constant \mathcal{L}_m is the Lagrangian of a matter. We consider a perfect fluid with the equation of state $p = \omega\rho$ as a matter source. The spacetime metric is

$$ds^2 = -dt^2 + \sum_{k=1}^N e^{2H_k t} dx_k^2, \quad H_k \equiv \text{const} \quad (2)$$

The dynamical equations, constraint and continuity equation are

$$2 \sum_{i \neq j} H_i^2 + 2 \sum_{\{i > k\} \neq j} H_i H_k + 8\alpha \sum_{i \neq j} H_i^2 \sum_{\{k > l\} \neq \{i, j\}} H_k H_l + 24\alpha \sum_{\{i > k > l > m\} \neq j} H_i H_k H_l H_m = -\omega \mathcal{H}, \quad j = \overline{1, N} \quad (3)$$

¹ Throughout the paper we use the system of units in which $G = c = 1$, G is the $(N + 1)$ -dimensional gravitational constant, c is the speed of light. Greek indices run from 0 to N , Latin indices from 1 to N .

$$2 \sum_{i>j} H_i H_j + 24\alpha \sum_{i>j>k>l} H_i H_j H_k H_l = \varkappa. \quad (4)$$

$$\dot{\rho} + (\rho + p) \sum_i H_i = 0 \quad (5)$$

where $\varkappa = 8\pi\rho$. Subtracting i -th dynamical equation from j -th one we obtain:

$$(H_j - H_i) \left(\frac{1}{4\alpha} + \sum_{\{k>l\} \neq \{i,j\}} H_k H_l \right) \sum_k H_k = 0 \iff \begin{cases} H_i = H_j & \text{(i)} \\ \sum_{\{k>l\} \neq \{i,j\}} H_k H_l = -\frac{1}{4\alpha} & \text{(ii)} \\ \sum_k H_k = 0 & \text{(iii)} \end{cases} \quad (6)$$

Left hand sides of Eqs. (3)–(4) does not depend on time, therefore $\rho \equiv \text{const}$, so that the Eq. (5) reduces to

$$(\rho + p) \sum_i H_i = 0 \iff \begin{cases} \rho = 0 & \text{(a)} \\ p = -\rho & \text{(b)} \\ \sum_k H_k = 0 & \text{(c)} \end{cases} \quad (7)$$

For a given set H_1, \dots, H_N to be a solution of Eqs. (3)–(4) it is necessary that at least one of the conditions **(i)**–**(iii)** is satisfied. In our previous work [26] we considered situations when conditions **(i)**, **(ii)** and their combinations are satisfied; it was found that taking into account these conditions lead to consistent system of equations for the vacuum **(a)** and Λ -term **(b)** cases only. In the present manuscript we interest in the condition **(iii)** and its combination with conditions **(i)**, **(ii)**. Requirement $\sum_k H_k = 0$ does not impose any constraints on choice of a matter from the continuity equation; we will see later that additional constraints on the form of the perfect fluid is followed from equations for the gravitational field. Note that taking into account condition $\sum_k H_k = 0$ one can rewrite Eqs. (3), (4) in the equivalent form:

$$\rho = -\frac{4 \left(\sum_i H_i^2 + \sum_{i>j} H_i H_j \right)}{3\omega - 1}, \quad \alpha = \frac{\rho(\omega - 1)}{16 \left(\sum_i H_i \sum_{i>j>k} H_i H_j H_k - \sum_{i>j>k>l} H_i H_j H_k H_l \right)} \quad (8)$$

III. CONSTANT VOLUME SOLUTIONS WITH TWO DIFFERENT HUBBLE PARAMETERS.

In the present section we generalize solutions found in [26] to an arbitrary equation of state of the matter. Taking into account results of the cited paper we assume that there are only two different parameters in the set H_1, \dots, H_N . Now one can easily obtain a number of special exact constant volume solutions. We consider three basic cases which are of great importance for low-dimensional spacetimes with $N = 4, 5$.

1. $([N-1]+1)$ -decomposition: $H_1 = \dots = H_{N-1} \equiv H \in \mathbb{R}$, $H_N \equiv h \in \mathbb{R}$. It follows from the condition $\sum_i H_i = 0$ that $h = -(N-1)H$. Substituting these H_1, \dots, H_N into Eqs. (16) we obtain:

$$\begin{cases} N(N-1)H^2 = -3\left(\omega - \frac{1}{3}\right)\left(\frac{\kappa}{2}\right) \\ 2N(N-1)(N^2-3N+3)H^4 = 9\left(\omega - \frac{1}{3}\right)^2\left(\frac{\kappa}{2}\right)^2 + \frac{\omega-1}{\alpha}\left(\frac{\kappa}{2}\right) \end{cases} \quad (9)$$

Solution of Eqs. (9) for H^2 and ρ :

$$H^2 = -\frac{1}{3\alpha(N-2)(N-3)}\frac{\omega-1}{\omega-\frac{1}{3}}, \quad \rho = \frac{1}{36\pi\alpha}\frac{N(N-1)}{(N-2)(N-3)}\frac{\omega-1}{\left(\omega-\frac{1}{3}\right)^2}, \quad \omega < \frac{1}{3}, \quad \alpha < 0 \quad (10)$$

2. $\left(\frac{N}{2} + \frac{N}{2}\right)$ -decomposition, N is even: $H_1 = \dots = H_{\frac{N}{2}} \equiv H \in \mathbb{R}$, $H_{\frac{N}{2}+1} = \dots = H_N \equiv h \in \mathbb{R}$. It follows from the condition $\sum_i H_i = 0$ that $h = -H$. Substituting these H_1, \dots, H_N into Eqs. (16) we obtain:

$$\begin{cases} NH^2 = -3\left(\omega - \frac{1}{3}\right)\left(\frac{\kappa}{2}\right) \\ 2NH^4 = 9\left(\omega - \frac{1}{3}\right)^2\left(\frac{\kappa}{2}\right)^2 + \frac{\omega-1}{\alpha}\left(\frac{\kappa}{2}\right) \end{cases} \quad (11)$$

Solution of Eqs. (11) for H^2 and ρ :

$$H^2 = \frac{1}{3\alpha(N-2)}\frac{\omega-1}{\omega-\frac{1}{3}}, \quad \rho = -\frac{1}{36\pi\alpha}\frac{N}{N-2}\frac{\omega-1}{\left(\omega-\frac{1}{3}\right)^2}, \quad \omega < \frac{1}{3}, \quad \alpha > 0 \quad (12)$$

3. $([n+1]+n)$ -decomposition, $n \equiv \left\lfloor \frac{N}{2} \right\rfloor$, N is odd: $H_1 = \dots = H_{n+1} \equiv H \in \mathbb{R}$, $H_{n+2} = \dots = H_N \equiv h \in \mathbb{R}$. It follows from the condition $\sum_i H_i = 0$ that $h = -(1+n^{-1})H$. Substituting these H_1, \dots, H_N into Eqs. (16) we obtain:

$$\begin{cases} N(1+n^{-1})H^2 = -3\left(\omega - \frac{1}{3}\right)\left(\frac{\kappa}{2}\right) \\ 2N(1+n^{-1})(1+n^{-1}+n^{-2})H^4 = 9\left(\omega - \frac{1}{3}\right)^2\left(\frac{\kappa}{2}\right)^2 + \frac{\omega-1}{\alpha}\left(\frac{\kappa}{2}\right) \end{cases} \quad (13)$$

Taking into account $n = \frac{N-1}{2}$ we get solution of Eqs. (11) for H^2 and ρ :

$$H^2 = \frac{1}{3\alpha}\frac{(N-1)^2}{(N-3)(N^2+N+2)}\frac{\omega-1}{\omega-\frac{1}{3}}, \quad \rho = -\frac{1}{36\pi\alpha}\frac{N(N-1)(N+1)}{(N-3)(N^2+N+2)}\frac{\omega-1}{\left(\omega-\frac{1}{3}\right)^2}, \quad (14)$$

$$\omega < \frac{1}{3}, \quad \alpha > 0$$

For $N = 4$ only cases 1 and 2 are realized: it is (3+1)-decomposition and (2+2)-decomposition; for $N = 5$ only cases 1 and 3 are realized: it is (4+1)-decomposition and (3+2)-decomposition and there are no other

options for $N = 4, 5$. It is easy to check that for $N = 4, 5$ and $\omega = -1$ solutions (10),(12),(14) turn to solutions derived in our previous paper [26] with additionally imposed constant volume requirement $\sum_i H_i = 0$.

For $N = 5$ there is one more decomposition, containing 3 different Hubble parameters (see [26]): $H_1 = H_2 \equiv H$, $H_3 = H_4 \equiv -H$, $H_5 \equiv h$; but it follows from $\sum_i H_i = 0$ that $h = 0$ and it reduces to $(2+2)$ -decomposition; for the same reasons decomposition $H_1 = \dots = H_n \equiv H \in \mathbb{R}$, $H_{n+1} = \dots = H_{2n} \equiv -H$, $H_N \equiv h \in \mathbb{R}$ ($n \equiv \lfloor \frac{N}{2} \rfloor$, N is odd) reduces to $(n+n)$ -decomposition described above. So, all possible generalization of solutions in $(4+1)$ and $(5+1)$ dimensions found in [26] for $w = -1$ to an arbitrary w are presented in our list.

We should note that in a general set-up (see the next section) other decompositions (for example, $(2+1+1)$ in $(4+1)$ dimensions) are possible and can be found by inserting corresponding ansatz into Eqs.(8). However, such solutions represent special cases of general solution with constant volume, and, unlike written down above, have no connections with varying volume solutions found in [26].

IV. NECESSARY CONDITIONS FOR GENERAL CONSTANT VOLUME SOLUTIONS.

In general case of constant volume solution we do not expect any additional relations between Hubble parameters (in contrast to the varying volume case, where only space-times with isotropic subspaces are possible). The full set of solution is rather cumbersome to be written down explicitly, so we restrict ourselves by finding conditions of its existence.

It was shown [25] that under assumption $\sum_i H_i = 0$ system (3)–(4) becomes

$$\begin{cases} -\sum_i H_i^2 + \alpha \left(\left[\sum_{j_1} H_{j_1}^2 \right]^2 - 2 \sum_i H_i^4 \right) = \omega \varkappa \\ \sum_i H_i^2 - 3\alpha \left(\left[\sum_{j_1} H_{j_1}^2 \right]^2 - 2 \sum_i H_i^4 \right) = -\varkappa \end{cases} \quad (15)$$

One can see that in the vacuum case ($\varkappa = 0$) the system (15) has no nontrivial² solution, except for the situation of pure Gauss-Bonnet model (the first term in both Eqs. (15) is absent) – the corresponding solution was found by [20]. It naturally follows from (15) that

$$\sum_i H_i^2 = -3 \left(\omega - \frac{1}{3} \right) \left(\frac{\varkappa}{2} \right), \quad \sum_i H_i^4 = \frac{1}{2} \left[9 \left(\omega - \frac{1}{3} \right)^2 \left(\frac{\varkappa}{2} \right)^2 + \frac{\omega - 1}{\alpha} \left(\frac{\varkappa}{2} \right) \right] \quad (16)$$

Obviously, for the system (16) to have nontrivial solutions it is necessary that

$$\omega - \frac{1}{3} < 0, \quad 9 \left(\omega - \frac{1}{3} \right)^2 \left(\frac{\varkappa}{2} \right)^2 + \frac{\omega - 1}{\alpha} \left(\frac{\varkappa}{2} \right) > 0 \quad (17)$$

² We call a solution trivial if $H_1 = \dots = H_N = 0$.

We see, first of all, that the equation of state parameter w is restricted from above: $w < 1/3$. However, positivity of quadratic and quartic sums is not sufficient for the solution to exist. Going further we denote:

$$\xi_1 = \frac{\varkappa}{2}, \quad \xi_2 = \frac{1}{\alpha} \left(\frac{\varkappa}{2} \right), \quad \xi = \frac{|\xi_2|}{\xi_1^2} \quad (18)$$

$$a = \xi_1(1 - 3\omega), \quad r^2 = \frac{1}{2} \left[\xi_1^2(1 - 3\omega)^2 + \xi_2(\omega - 1) \right], \quad \eta_k = H_k^2 \quad (19)$$

Then equations (16) take the form:

$$\sum_i \eta_i = a, \quad \sum_i \eta_i^2 = r^2 \quad (20)$$

Variables η_1, \dots, η_N can be considered as Cartesian coordinates in N -dimensional Euclidean space; then the first of the equations (20) specifies $(N - 1)$ -dimensional hyperplane which intersects each axis of the coordinate system at the point a , the second of the equations (20) describes $(N - 1)$ -dimensional hypersphere of radius r centred at the origin. Since $a > 0$ and all $\eta_i > 0$ we deal with fragments of the hypersphere and the hyperplane located in the first orthant. These fragments are intersected iff $r \leq a \leq \sqrt{Nr}$. Fig. 1 illustrates this reasoning for the 3D case. Obviously,

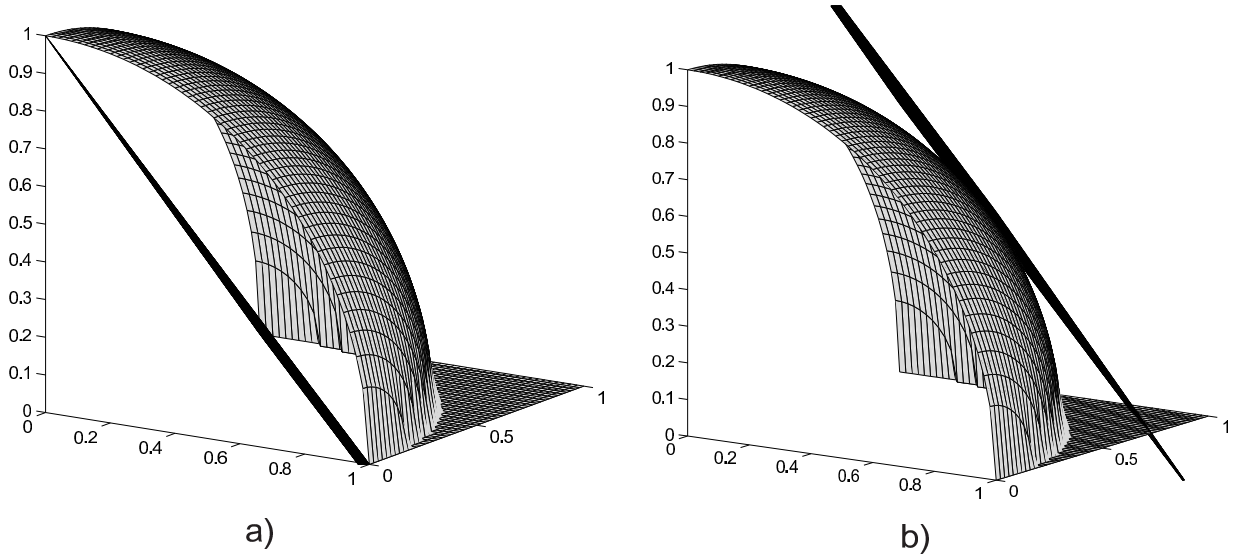


Figure 1. The plane and the sphere are intersected when the plane is placed between the two limiting positions shown on the pictures a) and b). The plane on the figure b) is tangent to the sphere.

$$\begin{cases} r \leq a \leq \sqrt{Nr} \\ a > 0 \end{cases} \iff \frac{1}{N} \leq \frac{r^2}{a^2} \leq 1 \quad (21)$$

So, system (16) has nontrivial solutions iff $\frac{r^2}{a^2} \in \left[\frac{1}{N}; 1 \right]$. We are concerned in such solutions of the system (16) that satisfy the condition $\sum_i H_i = 0$. It turns out that there is essential difference between even- and odd-dimensional cases. Indeed, let us consider $(4 + 1)$ -dimensional spacetime; Eqs. (20) describe 4-plane and

4-sphere; in the point of contact of these surfaces we have $H_1^2 = H_2^2 = H_3^2 = H_4^2$, therefore, one can choose H_1, \dots, H_4 such that $H_1 = H_2 = -H_3 = -H_4$ and the condition $H_1 + \dots + H_4 = 0$ is satisfied automatically. Clearly, there is no way one can satisfy the condition $H_1 + \dots + H_5 = 0$ in the point of tangency of 5-plane and 5-sphere because of one extra positive (or negative) summand. This results can be generalized to the case of arbitrary dimension: for an even-dimensional spacetime there exist solutions of the equations $\sum_i H_i^2 = a$, $\sum_i H_i^4 = r^2$ such that $\sum_i H_i = 0$ in the vicinity of the point of contact hyperplane and hypersphere specified by Eqs. (20); for an odd-dimensional spacetime there are no solutions in the vicinity the aforementioned point of contact. In general, there exist a subset $I \subseteq \left[\frac{1}{N}; 1\right]$ such that

$$\sum_i H_i = 0, \quad \sum_i H_i^2 = a, \quad \sum_i H_i^4 = r^2 \quad \text{for all} \quad \frac{r^2}{a^2} \in I \quad (22)$$

We express one of the Hubble parameters from the first of Eqs. (22) and substitute it in the remaining equations. Hubble parameters can be considered here as a Cartesian coordinates and lhs of the second of Eqs. (22) can be considered as a quadratic form; we reduce the latter to the canonical form by a coordinate transformation and after that convert a Cartesian coordinates to spherical $(\rho, \theta_1, \dots, \theta_{N-2})$. The second and the third equations then take the form $\rho^2 = a$, $\rho^4 f(\theta_1, \dots, \theta_{N-2}) = r^2$, where f is a polynomial in $\sin(\theta_k), \cos(\theta_k)$ for $k = \overline{1, N-2}$. Substituting $\rho^2 = a$ into the second of these equations we obtain $F(\theta_1, \dots, \theta_{N-2}, r, a) = 0$, where $F(\theta_1, \dots, \theta_{N-2}, r, a) = f(\theta_1, \dots, \theta_{N-2}) - \frac{r^2}{a^2}$. For example, for $N = 4$ we have:

$$\begin{aligned} F(\theta_1, \theta_2, r, a) = & \frac{1}{4} \sin^4(\theta_1) \left(\frac{1}{2} + \frac{11}{6} \cos^4(2\theta_2) \right) + \frac{1}{2} \left(\cos^2(\theta_1) + \sin^2(\theta_1) \sin^2(2\theta_2) \right)^2 + \\ & + \frac{1}{8} \sin^2(2\theta_1) \cos^2(2\theta_2) + \frac{1}{3\sqrt{2}} \sin(2\theta_1) \cos(2\theta_2) \left(\cos^2(\theta_1) - 3 \sin^2(\theta_1) \sin^2(2\theta_2) \right) - \frac{r^2}{a^2} \end{aligned} \quad (23)$$

So, the problem of the existence of solutions of Eqs. (22) is reduced to the problem of the existence of zeros of function F . We solve this problem numerically. Numerical calculations performed for $N \in \{4, \dots, 8\}$ shows that functions F has zeros for $\frac{r^2}{a^2} \in [\sigma_+; \sigma_-]$, i.e $I = [\sigma_+; \sigma_-]$ such that $\frac{1}{N} \leq \sigma_+ < \frac{1}{2} < \sigma_- < 1$. Using this fact we get:

$$\sigma_+ \leq \frac{r^2}{a^2} \leq \sigma_- \iff \begin{cases} [2\sigma_+ - 1] \xi_1^2 (1 - 3\omega)^2 \leq \xi_2 (\omega - 1) \\ [2\sigma_- - 1] \xi_1^2 (1 - 3\omega)^2 \geq \xi_2 (\omega - 1) \end{cases} \quad (24)$$

There are two cases, depending on the sign of the parameter α .

I. $\alpha > 0$.

$$\begin{cases} [2\sigma_+ - 1] \xi_1^2 (1 - 3\omega)^2 \leq \xi_2 (\omega - 1) \\ [2\sigma_- - 1] \xi_1^2 (1 - 3\omega)^2 \geq \xi_2 (\omega - 1) \end{cases} \iff \omega \leq \frac{1}{3} - \frac{\xi_+ + \sqrt{\xi_+ (\xi_+ + 24)}}{18}, \quad \xi_+ = \frac{\xi}{|2\sigma_+ - 1|} \quad (25)$$

II. $\alpha < 0$.

$$\begin{cases} [2\sigma_+ - 1] \xi_1^2 (1 - 3\omega)^2 \leq \xi_2 (\omega - 1) \\ [2\sigma_- - 1] \xi_1^2 (1 - 3\omega)^2 \geq \xi_2 (\omega - 1) \end{cases} \iff \omega \leq \frac{1}{3} - \frac{\xi_- + \sqrt{\xi_- (\xi_- + 24)}}{18}, \quad \xi_- = \frac{\xi}{2\sigma_- - 1} \quad (26)$$

Combining (25)-(26) with (??)-(??) we obtain finally:

$$\begin{cases} \sum_i H_i = 0 \\ \sum_i H_i^2 = -3 \left(\omega - \frac{1}{3} \right) \left(\frac{\kappa}{2} \right) \\ \sum_i H_i^4 = \frac{1}{2} \left[9 \left(\omega - \frac{1}{3} \right)^2 \left(\frac{\kappa}{2} \right)^2 + \frac{\omega - 1}{\alpha} \left(\frac{\kappa}{2} \right) \right] \end{cases} \iff \omega \leq \begin{cases} \frac{1}{3} - \frac{\xi_+ + \sqrt{\xi_+(\xi_+ + 24)}}{18}, \alpha > 0 \\ \frac{1}{3} - \frac{\xi_- + \sqrt{\xi_-(\xi_- + 24)}}{18}, \alpha < 0 \end{cases} \quad (27)$$

Inequalities (27) can be rewritten in terms of the energy density ρ :

$$\rho \geq \rho_{\text{lim}}(\omega), \quad \rho_{\text{lim}}(\omega) = \begin{cases} \frac{1}{36\pi\alpha} \frac{1}{2\sigma_+ - 1} \frac{\omega - 1}{\left(\omega - \frac{1}{3} \right)^2}, \alpha > 0 \\ \frac{1}{36\pi\alpha} \frac{1}{2\sigma_- - 1} \frac{\omega - 1}{\left(\omega - \frac{1}{3} \right)^2}, \alpha < 0 \end{cases} \quad (28)$$

We see that the above mentioned nonexistence of vacuum solutions has a sharper form: for any ω there exists a low limit for ρ . In the particular case of cosmological constant $\omega = -1$:

$$\rho_{\text{lim}}(-1) = \begin{cases} -\frac{1}{32\pi\alpha} \frac{1}{2\sigma_+ - 1}, \alpha > 0 \\ -\frac{1}{32\pi\alpha} \frac{1}{2\sigma_- - 1}, \alpha < 0 \end{cases} \quad (29)$$

Since the function $\rho_{\text{lim}}(\omega)$ is growing, for any non-fantom ($\omega \geq -1$) matter we have $\rho \geq \rho_{\text{lim}}(-1)$. Let us discuss briefly the problem of finding of the parameters σ_+, σ_- . We consider the cases of an even-dimensional and an odd-dimensional space separately.

I. It is easy to show that $\sigma_+ = \frac{1}{N}$ when the number N of space dimensions is even. Indeed, let N be an even number, in this case one can satisfy the condition $\sum_i H_i = 0$ by choosing parameters H_1, \dots, H_N such that

$$H_1 = -H_2, \dots, H_{N-1} = -H_N \quad (30)$$

It follows from (30) that $\eta_1 = \eta_2, \dots, \eta_{N-1} = \eta_N$, and equations (20) take the form:

$$\sum_{i=1}^{N/2} \eta_i = \frac{a}{2}, \quad \sum_{i=1}^{N/2} \eta_i^2 = \frac{r^2}{2}, \quad \eta_i > 0 \quad (31)$$

Repeating the above arguments we deduce that system (31) has nontrivial solutions iff

$$\frac{r}{\sqrt{2}} \leq \frac{a}{2} \leq \frac{r}{\sqrt{2}} \sqrt{\frac{N}{2}} \iff \frac{1}{N} \leq \frac{r^2}{a^2} \leq \frac{1}{2} \quad (32)$$

We see that $\sigma_+ = \frac{1}{N}$; substitution it into (28) leads to

$$\rho \geq \rho_{\text{lim}}(\omega) = -\frac{1}{36\pi\alpha} \frac{N}{N-2} \frac{\omega - 1}{\left(\omega - \frac{1}{3} \right)^2}, \quad \alpha > 0 \quad (33)$$

The case $\rho = \rho_{\text{lim}}$ corresponds exactly to the situation when plane $\sum_{i=1}^N \eta_i = a$ touches sphere $\sum_{i=1}^N \eta_i^2 = r^2$ and $\eta_1 = \eta_2, \dots, \eta_{N-1} = \eta_N$; in view of the condition $\sum_i H_i = 0$ the latter implies (30). In the previous section we described this case as $\left(\frac{N}{2} + \frac{N}{2}\right)$ -decomposition; it is easy to check that ρ_{lim} in (33) matches with ρ given in (12).

The upper threshold σ_- is a bit large than $\frac{1}{2}$ (condition (30) is not necessary: there exist other sets of the Hubble parameters such that $\sum_i H_i = 0$). For $N = 4$ numerical calculations give $\sigma_- = 0.76 \pm 0.01$. So, for $(4 + 1)$ -dimensional spacetime we have:

$$\omega < \begin{cases} \frac{1}{3} - \frac{2\xi + \sqrt{2\xi + (2\xi + 24)}}{18}, & \alpha > 0 \\ \frac{1}{3} - \frac{1.92\xi + \sqrt{1.92\xi(1.92\xi + 24)}}{18}, & \alpha < 0 \end{cases}, \quad \text{or} \quad \rho \gtrsim \begin{cases} \frac{1}{18\pi\alpha} \frac{\omega - 1}{\left(\omega - \frac{1}{3}\right)^2}, & \alpha > 0 \\ -\frac{1}{1.04} \frac{1}{18\pi\alpha} \frac{\omega - 1}{\left(\omega - \frac{1}{3}\right)^2}, & \alpha < 0 \end{cases} \quad (34)$$

where $\xi = \frac{1}{4\pi|\alpha|\rho}$.

II. When working with an odd-dimensional space we have no possibility to satisfy the condition $\sum_i H_i = 0$ for $\sigma_+ = \frac{1}{N}$. Indeed, system (20) has the only solution $\eta_1 = \dots = \eta_N = \frac{r}{\sqrt{N}}$ for $\frac{r^2}{a^2} = \frac{1}{N}$ (geometrically it corresponds to point of contact plane and sphere – see Fig. 1a). Since $H_1 = \dots = H_N = \pm \sqrt{\frac{r}{\sqrt{N}}}$, the sum $\sum_i H_i$ has at least one extra positive (or negative) term and can not vanish for odd N . For $N = 5$ numerical calculations give $\sigma_+ = 0.23 \pm 0.01$, $\sigma_- = 0.65 \pm 0.01$. So, for $(5 + 1)$ -dimensional spacetime we have:

$$\omega < \begin{cases} \frac{1}{3} - \frac{1.85\xi + \sqrt{1.85\xi + (1.85\xi + 24)}}{18}, & \alpha > 0 \\ \frac{1}{3} - \frac{3.33\xi + \sqrt{3.33\xi(3.33\xi + 24)}}{18}, & \alpha < 0 \end{cases}, \quad \text{or} \quad \rho \gtrsim \begin{cases} \frac{1}{0.54} \frac{1}{36\pi\alpha} \frac{\omega - 1}{\left(\omega - \frac{1}{3}\right)^2}, & \alpha > 0 \\ -\frac{1}{0.3} \frac{1}{36\pi\alpha} \frac{\omega - 1}{\left(\omega - \frac{1}{3}\right)^2}, & \alpha < 0 \end{cases} \quad (35)$$

where $\xi = \frac{1}{4\pi|\alpha|\rho}$.

V. CONCLUSIONS

In the present paper we have considered solutions with constant different Hubble parameters in a flat Einstein-Gauss-Bonnet cosmology. Such solutions are absent in a pure Einstein gravity and its existence represent one of specific features of higher-order curvature terms (see [24] for details).

Combining results of the present paper with results of our previous paper [26] we can write down full classification of solutions in question in $(4+1)$ and $(5+1)$ dimensions.

- Vacuum solution in a pure Gauss-Bonnet gravity [20]. We have shown that this solution is a particular one and can not be incorporated in other sets of solution of the type considered. It requires absence of both matter and Einstein-Hilbert term.

- Solutions with volume element changing in time. Such solutions require a matter only in the form of cosmological constant. Apart from an isotropic solution, it appears that these solutions exist only when set of Hubble parameters is divided into subsets with equal values of Hubble parameters belonging to the same subset (so, existence of isotropic subspaces is required).
- Solutions with constant volume element. Solutions of this type exist only when matter density exceeds (or equal to) some critical value which depends on the equation of state of the matter. The parameter ω of the matter should be smaller than $1/3$. In general, solutions do not have isotropic subspaces, though can have them for special cases.

As space-times with isotropic subspaces represent a particular interest (for example, if multidimensional paradigm is indeed realized in Nature, then our own world belongs to this class) we write down explicit solutions of constant volume element with isotropic subspaces, generalising those found in [26]. For a general case of constant volume element (without isotropic subspaces) we present the conditions for such solutions to exist, leaving their explicit form to a future work.

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